Superconductor–metal transition in an ultrasmall Josephson junction biased by a noisy voltage source

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Shot noise in a voltage source changes the character of the quantum (dissipative) phase transition in an ultrasmall Josephson junction: The *superconductor–insulator* transition transforms into the *superconductor–metal* transition. In the metallic phase the IV curve probes the voltage distribution generated by shot noise, whereas in the superconducting phase it probes the counting statistics of electrons traversing the noise junction.

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The phenomenon of noise is continuing to be in the focus of modern mesoscopic physics. In particular, a lot of attention is devoted to shot noise [1], which is related to discreteness of charge transport and yields direct information on carrier charges. Efforts of theorists were invested to studying full counting statistics of shot noise, its non-Gaussian character, and asymmetry (odd moments) [2, 3, 4]. They are also objects of intensive experimental investigations [5, 6]. This stimulated development of effective methods of noise detection [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. It has been shown theoretically and experimentally that Coulomb blockade of an ultrasmall Josephson junction is very sensitive to noise from an independent source [8, 10, 11, 15]. This can be used for noise spectroscopy. In the experiment they studied the effect of shot noise on the low-voltage biased Josephson junction [8, 10]. The source of shot noise was a current through another junction parallel to the Josephson junction. The current from the additional (noise) junction generated the voltage drop on the shunt resistance, which was added to the voltage bias on the Josephson junction. Originally this phenomenon was investigated al low noise currents, when electron tunneling events produced a sequence of voltage pulses on the shunt resistance well separated in time [8, 10, 11]. Later on the theoretical analysis was expended on arbitrary high noise currents [15]. The analysis has shown that whatever high the current through noise junction is, the voltage drop produced by the noise current cannot be considered as an ideal voltage bias. Thus the response of the Josephson junction to this "noisy" voltage source is essentially different from that to the ideal source of constant voltage.

This Letter investigates this interesting phenomenon. The analysis addresses the general case when the Josephson junction is biased both with the ideal and the noisy voltage source, but the focus is on the case, when the current through the additional junction is the *only* source of the voltage bias on the Josephson junction. The response of the Josephson junction to this "noisy" voltage source is different from that to the ideal (constant) voltage source in many aspects. However, the most interesting outcome is that the well known "superconductor–
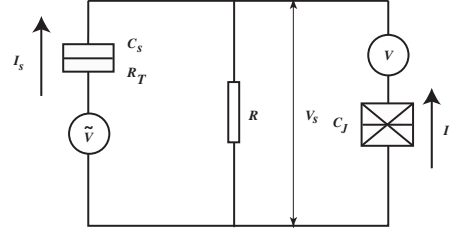

FIG. 1: Electric circuit with two sources of constant voltages V and $\tilde{V}$. The Josephson junction with the capacitance C_J is biased with the voltage V and the fluctuating voltage drop V_s at the shunt. The latter originates from the average current $I_s = \tilde{V}/R_T$ through the noise junction of capacitance C_s.

insulator" transition [17, 18, 19], which takes place at $\rho = R/R_Q = 1$, transforms to the "superconductor–metal" transition. Here $R_Q = h/4e^2 = \pi\hbar/2e^2$ is the quantum resistance for Cooper pairs. This means that at $\rho > 1$ the zero-bias conductance does not vanish, but remains finite as in a junction between two normal metals.

Figure 1 shows the electric circuit discussed in the paper. Two voltages can bias the Josephson junction of capacitance C_J: (i) the constant voltage V (ideal voltage bias), and (ii) the fluctuating voltage drop V_s at the shunt resistance R. The average voltage $\bar{V}_s = I_s R$ is determined by the average current I_s through the additional noise normal junction of capacitance C_s and resistance R_T. It is assumed that $R_T \gg R$. Then $I_s = \tilde{V}/R_T$, the noise junction is voltage biased and tunneling events at the junction are governed by the Poissonian statistics.

If only the ideal voltage bias is used ($I_s = 0$), the $P(E)$ theory of incoherent tunneling of Cooper pairs yields the following current through the Josephson junction [17, 18, 19]:

$$I = \frac{\pi e E_J^2}{\hbar}[P_0(2eV) - P_0(-2eV)] , \qquad (1)$$

where the $P(E)$ function

$$P_0(E) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \exp\left[J_0(t) + \frac{iEt}{\hbar}\right] , \qquad (2)$$

characterizes the probability to transfer the energy $E > 0$ to environment (or to absorb the energy $|E|$ from environment if $E < 0$). We restrict ourselves with the zero-temperature limit when the phase-phase correlator, which determines the $P(E)$ function, is given by [11, 15]

$$J_0(t) = \langle[\varphi_0(t) - \varphi_0(0)]\varphi_0(0)\rangle = \rho\left[-e^{t/\tau}E_1\left(\frac{t}{\tau}\right)\right.$$
$$\left. -e^{-t/\tau}E_1\left(-\frac{t}{\tau} + i0\right) - 2\ln\frac{t}{\tau} - 2\gamma - i\pi\right] , \quad (3)$$

where $\tau = RC$ is the relaxation time in the electric circuit, $C = C_J + C_s$, $\gamma = 0.577$ is the Euler constant, and $E_1(z) = \int_1^{\infty} e^{-zt} dt/t$ is the exponential integral. The $P(E)$ theory is based on the time-dependent perturbation theory with respect to the small Josephson coupling energy E_J and uses the Golden Rule for calculation of the tunneling probability. In addition, it is assumed that phase fluctuations are Gaussian. At $T = 0$ $P(E)$ vanishes for $E < 0$ since it is the probability of the transfer of the energy $|E|$ from the environment to the junction, which is impossible if $T = 0$. The subscript 0 points out that the phase fluctuations φ_0 and the $P_0(E)$ function are determined by the equilibrium Johnson-Nyquist noise.

If the voltage bias is not "ideal", i.e., the constant voltage V is supplemented with the fluctuating voltage drop V_s at the shunt resistance R, the $P_0(E)$ function in the expression for the current, Eq.(1), should be replaced by a more general function [11, 15]:

$$P(2eV) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt e^{J_0(t)} e^{i2eVt/\hbar} \langle\exp[i\Delta\varphi_s(t)]\rangle .$$
$$(4)$$

The phase difference $\Delta\varphi_s(t) = \varphi_s(t) - \varphi_s(0) = (2e/\hbar)\int_0^t V_s(t)$ is determined by the fluctuating voltage $V_s(t)$. Introducing the $P(E)$ function for shot noise,

$$P_s(E) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt e^{iEt/\hbar} \langle\exp[i\Delta\varphi_s(t)]\rangle , \qquad (5)$$

the total $P(E)$ function is determined by a convolution of the two $P(E)$ functions [20]:

$$P(2eV) = 2e \int_{-\infty}^{\infty} dV_1 P_0(2eV_1) P_s(2e[V - V_1]) . \qquad (6)$$

The averaged phase correlator in the expression for the shot-noise $P_s(E)$ function, Eq. (5), is a value of the generating function $\langle\exp[\xi\Delta\varphi_s(t)]\rangle$ at $\xi = i$. The generating function determines all moments and cumulants of the random phase difference $\Delta\varphi_s(t)$. For an ideal voltage bias of the noise junction this generating function can be determined exactly [15, 22] keeping in mind that the electron transport through the noise junction produces a sequence of random voltage pulses at the shunt resistance R:

$$V_s(t) = \text{sign}(I_s)(e/C) \sum_i \Theta(t - t_i) e^{-(t-t_i)/\tau} , \qquad (7)$$

where t_i are random moments of time when an electron crosses the junction. The average time interval between tunneling events is e/I_s, and the number of the events in a fixed time interval is governed by Poissonian statistics. This sequence of voltage pulses generates the sequence of phase jumps:

$$\varphi_s(t) = \text{sign}(I_s)\pi\rho \sum_i \Theta(t - t_i)\left[1 - e^{-(t-t_i)/\tau}\right] . \quad (8)$$

The generating function for the random phase difference $\Delta\varphi_s(t)$ is given by

$$\langle\exp[\xi\Delta\varphi_s(t)]\rangle = \exp\left[\frac{I_s\tau}{e}\Phi(\xi, t)\right] , \qquad (9)$$

where

$$\Phi(\xi, t) = -e^{\pi\rho\xi}\left[E_1(\pi\rho\xi) - E_1\left(\pi\rho\xi e^{-t/\tau}\right)\right] - \frac{t}{\tau} - \gamma$$
$$- \ln\left[-\pi\rho\xi\left(1 - e^{-t/\tau}\right)\right] - E_1\left[-\pi\rho\xi\left(1 - e^{-t/\tau}\right)\right] \quad (10)$$

In the high-impedance case $\rho \gg 1$, the main contribution to the time integral in Eq. (4) for the $P(E)$ function comes from times $t \sim RC/\rho = R_Q C$ much shorter than $\tau = RC$ [see the analytic calculation of the ratchet current, Eq. (34), in Ref. 15]. Then the voltage does not vary essentially during the time interval t, i.e., $\Delta\varphi_s(t) \approx 2eV_s t/\hbar$, and the expression for $\Phi(\xi, t)$ can be simplified:

$$\Phi(\xi, t) = -E_1(-\pi\rho\xi t/\tau) - \gamma - \ln(-\pi\rho\xi t/\tau) . \quad (11)$$

This means that the full statistics of the phase difference is identical to the full statistics of voltage fluctuations as

found in Ref. 22. Indeed, for the sequence of random voltage pulses giving by Eq. (7), the generating function for voltage probability at the shunt is:

$$F(\nu) = \langle \exp[\nu V_s C/e] \rangle = \exp\left[\frac{I_s \tau}{e} \Phi_v(\nu)\right] , \quad (12)$$

where

$$\Phi_v(\nu) = -E_1(-\nu) - \gamma - \ln(-\nu) . \quad (13)$$

One can see that $\Phi(\xi, t)$ given by Eq. (11) is identical to $\Phi_v(\nu)$ in Eq. (13) with $\nu = \pi\rho\xi t/\tau$. Altogether this means that the voltage distribution generated by shot noise,

$$p(V_s) = \frac{C}{2\pi e} \int_{-\infty}^{\infty} dx e^{-ixV_s C/e} F(ix) , \quad (14)$$

directly determines the shot-noise $P(E)$ function:

$$P_s(E) = \frac{1}{2e} p(-E/2e) . \quad (15)$$

Thus the total $P(E)$ function is the $P(E)$ function for the equilibrium noise averaged over the voltage distribution generated by the noise current:

$$P(2eV) = \int_{-\infty}^{\infty} dV_1 P_0(2eV_1) p(V_1 - V) . \quad (16)$$

This approach called "the time-dependent $P(E)$ theory" has already been used in previous numerical simulations [21]. The present analysis has justified this approach for the high-impedance environment $\rho \gg 1$, when the $P(E)$ function (current) probes the voltage statistics. But the approach is not valid in the opposite case of the low-impedance environment $\rho < 1$ (see below).

In the high-impedance limit $\rho = R/R_Q \to \infty$ the equilibrium $P(E)$ function can be approximated with the δ-function peak: $P_0(2eV) = \delta(2eV - 2e^2/C)$ [23]. Then according to Eq. (16) the $P(E)$ function (current) directly scans the voltage probability distribution generated by shot noise and given by Eq. (14):

$$P(2eV) = \frac{1}{2e} p\left(\frac{e}{C} - V\right) . \quad (17)$$

The dependence of this function on $\bar{V}_s$ is plotted in Fig. 2 for the case when the Josephson junction is biased only with noisy voltage drop ($V = 0$). In contrast to the ideal voltage bias, which in the limit $\rho \to \infty$ yields the sharp peak $P(2eV) = (1/2e)\delta(V - e/C)$, the noisy voltage bias yields the non-Gaussian broad maximum.

While the approximation based on Eq. (17) is quite satisfactory for calculation of the current dependence on noisy voltage bias V_s, it is not sufficient for description of the effect of weak shot noise ($\bar{V}_s \ll e/C$) on dependence on ideal voltage bias V, which was studied in Refs. 11, 15.
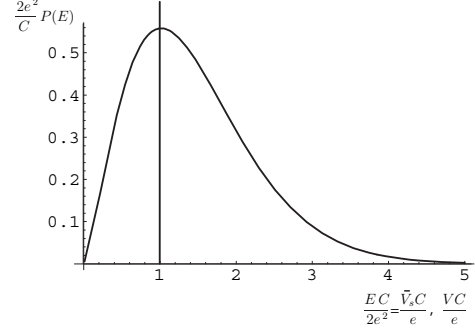

FIG. 2: $P(E)$ function in the limit of high impedance $R \gg R_Q$ for the ideal voltage bias ($E = 2eV$, the δ-function peak at $V = e/C$) and for the noisy voltage bias ($E = 2eV_s$).

In order to show it we linearize Eq. (14) with respect to $\bar{V}_s$. Taking the integral by parts one obtain:

$$p(V_s) = \frac{\bar{V}_s C^2}{2\pi e^2} \int_{-i\infty}^{i\infty} d\nu e^{-\nu V_s C/e} \Phi_v(\nu)$$

$$= \frac{\bar{V}_s C}{2\pi e V_s} \int_{-\infty}^{\infty} e^{-ixV_s C/e} \frac{e^{ix} - 1}{ix} dx$$

$$= \frac{\bar{V}_s C}{2\pi e V_s} \int_{-\infty}^{\infty} \frac{\sin[x(1 - V_s C/e)] + \sin(xV_s C/e)}{x} dx . (18)$$

This yields $p(V_s) = \bar{V}_s C/eV_s$ at $0 < V_s < e/C$ and $p(V_s) = \bar{V}_s C/2eV_s$ if $V_s = e/C$ exactly. Otherwise ($V_s < 0$ or $V_s > e/C$) the voltage probability vanishes. The obtained probability density is the voltage distribution inside a single voltage pulse $V_s(t) = (e/C)\exp(-t/\tau)$, and $p(V_s) \propto dt/dV_s$. Inserting this voltage distribution into Eq. (17) one obtains the $P(E)$ function plotted as a function of the ideal voltage bias V in Fig. 3. There are two singularities on this dependence: a divergence near $V = e/C$ and a jump near the origin $V = 0$. But singularities are smeared out if one takes into account that the peak of $P_0(E)$ at $E = e^2/C$ is not a δ-function, but has a finite width $\sim e^2/C\sqrt{\rho}$ (ignoring a logarithm factor). This more accurate approach, which was used in Refs. 11 and 15, leads to smearing of the zero-voltage jump onto voltages of the order of $e/C\sqrt{\rho}$ and to a finite linear slope at $V = 0$ (the dashed line in Fig. 3). The linear slope corresponds to metallic conductance analytically calculated in Ref. 11 for $\rho \gg 1$ [see Eq. (20) there].

In the opposite limit of low-impedance environment $\rho \ll 1$ the most important contribution to the $P(E)$ function comes from long times $t \gg \tau$, and the $P(E)$ function does not scan the voltage distribution anymore. In this limit we need the asymptotic expression for the Johnson-Nyquist correlator, Eq. (3):

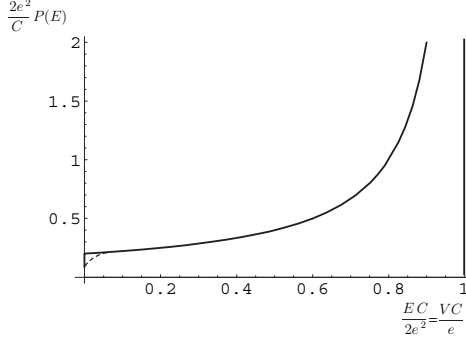$$J_0(t) = -2\rho \left(\ln\frac{t}{\tau} + \frac{i\pi}{2}\right) , \quad (19)$$

FIG. 3: $P(E)$ function at weak noise current ($\bar{V}_sC/e = 0.15$). The dashed line shows smearing of the zero-bias jump with finite width of the peak in the equilibrium $P_0(E)$ function.

and for the logarithm of the generating function for the full phase statistics, Eq. (10):

$$\Phi(\xi, t) = \frac{t}{\tau}(e^{\pi\rho\xi} - 1) . \qquad (20)$$

This expression corresponds to the Poissonian statistics of phase jumps, identical to the Poissonian full counting statistics [15, 22]. Since $\rho \ll 1$ we can expand in ρ. At the same time we can generalize this expansion on other possible types of statistics. Then the phase correlator (the generating function at $\xi = i$) can be written as

$$\langle\exp\left[i\Delta\varphi_s(t)\right]\rangle = \exp\left[\frac{i2e\bar{V}_s t}{\hbar}\lambda\right] , \qquad (21)$$

where

$$\lambda = \lambda_R + i\lambda_I \approx 1 + \frac{i\pi\rho\xi}{2}\frac{\langle\langle n^2\rangle\rangle}{\langle\langle n\rangle\rangle} - \frac{\pi^2\rho^2\xi^2}{6}\frac{\langle\langle n^3\rangle\rangle}{\langle\langle n\rangle\rangle} . \qquad (22)$$

Here $\langle\langle n^k\rangle\rangle$ are cumulants of the full counting statistics, n being a number of electrons traversing the noise junction during the time t. For the Poissonian statistics all cumulants are equal to the first cumulant $\langle\langle n\rangle\rangle$, which is the average number of electrons traversing the junction. Inserting the expression (21) into Eq. (4) one obtains the $P(E)$ function in the low-impedance limit:

$$P(2eV) = \frac{2}{\pi\hbar}\int_0^\infty dt \left(\frac{\tau}{t}\right)^{2\rho}\sin(\pi\rho)$$

$$\times\mathrm{Im}\left\{\exp\left[i\left(\frac{2eVt}{\hbar} + \frac{2e\bar{V}_s t}{\hbar}\lambda\right)\right]\right\}$$

$$\approx \frac{2\tau\rho}{\hbar}\left[\frac{\hbar}{2e\tau(V + \bar{V}_s|\lambda|)}\right]^{1-2\rho} . \qquad (23)$$

Though one cannot use this expression at very low voltages where the perturbation theory in E_J becomes invalid [24], the "superconductivity" current peak is present anyway, both for ideal and noisy voltage bias. Equation (23)

demonstrates that in low impedance environment the IV curve probes the counting statistics even though the dependence on high cumulants $k > 2$ is not so pronounced.

In summary, shot noise in the voltage source dramatically changes the character of the quantum (dissipative) phase transition in the ultrasmall Josephson junction tuned by the environment impedance. For the ideal voltage bias this is a transition from the superconducting state ($\rho = R/R_Q < 1$) to the insulator (Coulomb blockade, $\rho = R/R_Q > 1$). In contrast, in the case of the noisy voltage source the transition at $\rho = R/R_Q = 1$ is between the superconducting phase and the metallic phase with finite zero-bias conductance. This transition can be called *superconductor–metal transition*. In the metallic phase the IV curve is a probe of the voltage distribution generated by shot noise, whereas in the superconducting phase the IV curve is probing the counting statistics for electrons traversing the noise junction.

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